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Conjugate duality in vector optimization and some applications to the vector variational inequality

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Abstract

The aim of this paper is to extend the so-called perturbation approach in order to deal with conjugate duality for constrained vector optimization problems. To this end we use two conjugacy notions introduced in the past in the literature in the framework of set-valued optimization. As a particular case we consider a vector variational inequality which we rewrite in the form of a vector optimization problem. The conjugate vector duals introduced in the first part allow us to introduce new gap functions for the vector variational inequality. The properties in the definition of the gap functions are verified by using the weak and strong duality theorems.

Keywords: Conjugate duality; Conjugate map; Vector optimization; Perturbation function; Vector variational inequality; Gap function

1. Introduction

Having a scalar optimization problem one can attach to it a conjugate dual problem by using the so-called perturbation approach [7]. This topic has been intensively investigated in the past, the mentioned approach giving new insights in the duality theory. Boţ and Wanka considered for a primal scalar optimization problem three conjugate duals, namely the Lagrange, Fenchel and

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Fenchel–Lagrange dual problems, all of them obtained by considering some special perturbation functions [5,23]. The relations between the optimal objective functions of these duals have been completely investigated.

Inspired by the scalar case, Tanino and Sawaragi in [20] (see also [15]) developed a conjugate duality theory based on the same perturbation approach but for vector optimization problems. They considered a new concept of a conjugate map (also called *type I Fenchel transform*) in finite-dimensional spaces based on Pareto efficiency. For different perturbation functions one can attach to a vector optimization problem, by using this conjugacy notion, a dual problem. Weak and strong duality assertions can be also formulated. Furthermore, by using the concept of supremum of a set (cf. [21]) on the basis of weak orderings, this conjugate duality theory has been extended to optimization problems in partially ordered topological vector space (see [22]) and to set-valued vector optimization problems (see [19]).

In the first part of the paper we construct for a primal vector optimization problem, by considering some appropriate perturbation functions, three new vector dual problems based on the Lagrange, Fenchel and Fenchel–Lagrange duality concepts treated in [23] in the scalar case. To this end we consider on the one hand the type I Fenchel transform introduced in [15] but also, on the other hand, a different conjugacy concept, namely the *type II Fenchel transform*. For the definition and some property of the latter we mention the book of Goh and Yang [11]. For all the vector duals considered in this paper weak and strong duality assertions are proved.

In the second part of the paper we deal with the connections between vector optimization and vector variational inequalities. Since the vector variational inequality in a finite-dimensional space was introduced first in [8], several papers concerning this topic have been written in the past (see, for instance, [10,13,14]. By rewriting a vector variational inequality in the form of a vector optimization problem, the conjugate vector duals introduced in the first part allow us to introduce new gap (merit) functions for it.

In the case of scalar optimization the construction of a gap function for variational inequalities has been associated to Lagrange duality (see [9]). Different classes of gap (merit) functions have been considered also by Noor in [18] for general variational inequalities in Hilbert spaces. The general variational inequalities have been introduced in [16] and include as special cases variational inequalities, quasivariational inequalities and complementary problems (see also [17]). By using the gap functions introduced in [18] one can obtain error bounds for the solution of general variational inequalities.

The approach we consider here extends to the vector case the results in [2,3] where different gap functions have been constructed via conjugate duality for variational inequalities and for equilibrium problems, respectively.

The paper is organized as follows. In Section 2 we recall some definitions and some preliminary results. In Section 3 we develop a new duality theory for the constrained vector optimization problem by using the perturbation approach and working with the type I Fenchel transform. Similar results are obtained in Section 4 but using the type II Fenchel transform. These duality concepts allow us to define in Section 5 some new gap functions for the vector variational inequality. The properties in the definition of the gap functions are verified by using the weak and strong duality theorems.

2. Mathematical preliminaries

Let C be a pointed closed and convex cone in $\mathbb{R}^n$. For any $\xi, \mu \in \mathbb{R}^n$, we use the following ordering relations: $\xi \underset{C}{\leqslant} \mu \Leftrightarrow \mu - \xi \in C$, $\xi \underset{C\setminus\{0\}}{\leqslant} \mu \Leftrightarrow \mu - \xi \in C \setminus \{0\}$ and $\xi \underset{C\setminus\{0\}}{\nleqslant} \mu \Leftrightarrow \mu - \xi \notin C \setminus \{0\}$. The notions $\underset{C}{\geqslant}$, $\underset{C\setminus\{0\}}{\geqslant}$ and $\underset{C\setminus\{0\}}{\ngeqslant}$ are used in an alternative way.

Definition 2.1. A point $y \in \mathbb{R}^n$ is said to be a maximal point of a set $Y \subseteq \mathbb{R}^n$ if $y \in Y$ and there is no $y' \in Y$ such that $y \underset{C\setminus\{0\}}{\leqslant} y'$.

The set of all maximal points of Y is called the maximum of Y and is denoted by $\max_{C\setminus\{0\}} Y$. The minimum of Y is defined analogously. Further we take the cone C being the nonnegative orthant $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n)^T \in \mathbb{R}^n \mid x_i \geqslant 0,\ i = \overline{1, n}\}$.

Lemma 2.1. [15, cf. Proposition 3.1.3] *Let $Y_1, Y_2 \subseteq \mathbb{R}^n$. Then*

(i) $\max_{\mathbb{R}^n_+\setminus\{0\}}(Y_1 + Y_2) \subseteq \max_{\mathbb{R}^n_+\setminus\{0\}} Y_1 + \max_{\mathbb{R}^n_+\setminus\{0\}} Y_2$;
(ii) $\min_{\mathbb{R}^n_+\setminus\{0\}}(Y_1 + Y_2) \subseteq \min_{\mathbb{R}^n_+\setminus\{0\}} Y_1 + \min_{\mathbb{R}^n_+\setminus\{0\}} Y_2$.

Definition 2.2. [11, cf. Definition 8.2.2] Let $h : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map.

(i) The set-valued map $\min_{\mathbb{R}^p_+\setminus\{0\}} h(x)$ is said to be externally stable if $h(x) \subseteq \min_{\mathbb{R}^p_+\setminus\{0\}} h(x) + \mathbb{R}^p_+$, $\forall x \in \mathbb{R}^n$.
(ii) Similarly, the set-valued map $\max_{\mathbb{R}^p_+\setminus\{0\}} h(x)$ is said to be externally stable if $h(x) \subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} h(x) - \mathbb{R}^p_+$, $\forall x \in \mathbb{R}^n$.

Lemma 2.2. [15, Lemma 6.1.1] *Let $F_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $F_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be set-valued maps and $X \subseteq \mathbb{R}^n$. Then*

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \big[F_1(x) + F_2(x)\big] \subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big[F_1(x) + \max_{\mathbb{R}^p_+\setminus\{0\}} F_2(x)\Big].$$

If $\max_{\mathbb{R}^p_+\setminus\{0\}} F_2(x)$ *is externally stable, then the converse inclusion also holds.*

Corollary 2.1. [15, Corollary 6.1.3] *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map and $X \subseteq \mathbb{R}^n$. If* $\max_{\mathbb{R}^p_+\setminus\{0\}} F(x)$ *is externally stable, then*

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} F(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \max_{\mathbb{R}^p_+\setminus\{0\}} F(x).$$

Before describing the conjugate duality for vector optimization, let us recall the concepts of conjugate maps and of set-valued subgradient.

Definition 2.3. [11, Definition 8.2.1] Let $h : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map.

(i) The set-valued map $h^* : \mathbb{R}^{p\times n} \rightrightarrows \mathbb{R}^p$ defined by $h^*(U) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n}[Ux - h(x)]$, $U \in \mathbb{R}^{p\times n}$ is called the conjugate map of h.

(ii) The conjugate map of h^*, h^{**} is called the biconjugate map of h, i.e.

$$h^{**}(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{U\in\mathbb{R}^{p\times n}} \big[Ux - h^*(U)\big], \quad x \in \mathbb{R}^n.$$

(iii) U is said to be a subgradient of the set-valued map h at $(\bar{x}; \bar{y})$ if $\bar{y} \in h(\bar{x})$ and

$$\bar{y} - U\bar{x} \in \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} \big[h(x) - Ux\big].$$

The set of all subgradients of h at $(x; y)$ is denoted by $\partial h(x; y)$ and is called the *subdifferential of h at* $(x; y)$. If $\partial h(x; y) \neq \emptyset$, $\forall y \in h(x)$, then h is said to be *subdifferentiable at* x.

When $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ is a vector-valued function, then the conjugate map φ^* of φ is defined by

$$\varphi^*(T) = \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{Tx - \varphi(x) \bigm| x \in \mathbb{R}^n\big\}, \quad T \in \mathbb{R}^{p\times n}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^p \cup \{\infty\}$ be an extended vector-valued function. Here ∞ is the imaginary point whose every component is $+\infty$. We consider the following unconstrained vector optimization problem

$$(P_u) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{f(x) \bigm| x \in \mathbb{R}^n\big\}.$$

In other words, (P_u) is the problem of finding $\bar{x} \in \mathbb{R}^n$ such that $f(x) \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleq} f(\bar{x})$, $\forall x \in \mathbb{R}^n$. Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{\infty\}$ be another vector-valued function such that $\Phi(x, 0) = f(x)$, $\forall x \in \mathbb{R}^n$, which is the so-called *perturbation function*. *The value function* is a set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^p \cup \{\infty\}$ defined by $\Psi(y) = \min_{\mathbb{R}^p_+\setminus\{0\}}\{\Phi(x, y) \mid x \in \mathbb{R}^n\}$. Clearly $\Psi(0) = \min_{\mathbb{R}^p_+\setminus\{0\}} f(\mathbb{R}^n)$ is the minimal frontier of the problem (P_u). The problem (P_u) can be stated as the primal optimization problem

$$(P_u) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{\Phi(x, 0) \bigm| x \in \mathbb{R}^n\big\}.$$

The conjugate map of Φ, denoted by $\Phi^* : \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \rightrightarrows \mathbb{R}^p \cup \{\infty\}$, is a set-valued map defined in the usual manner: $\Phi^*(U, V) = \max_{\mathbb{R}^p_+\setminus\{0\}}\{Ux + Vy - \Phi(x, y) \mid x \in \mathbb{R}^n,\ y \in \mathbb{R}^m\}$. Then the conjugate dual optimization problem can be defined as being

$$(D_u) \qquad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{V\in\mathbb{R}^{p\times m}} \big[-\Phi^*(0, V)\big].$$

Since $-\Phi^*$ is a set-valued map, the problem (D_u) is not an ordinary vector optimization problem. In other words, it can be reformulated as follows:

$$\text{Find} \quad V^* \in \mathbb{R}^{p\times m} \quad \text{such that} \quad -\Phi^*\big(0, V^*\big) \cap \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{V\in\mathbb{R}^{p\times m}} \big[-\Phi^*(0, V)\big] \neq \emptyset.$$

Theorem 2.1 *(Weak duality).* [15, Proposition 6.1.12]

$$\Phi(x, 0) \notin -\Phi^*(0, V) - \mathbb{R}^p_+ \setminus \{0\}, \quad \forall x \in \mathbb{R}^n,\ \forall V \in \mathbb{R}^{p\times m}.$$

Definition 2.4. The primal problem (P_u) is said to be stable with respect to the perturbation function Φ if the value function Ψ is subdifferentiable at $y = 0$.

Theorem 2.2 *(Strong duality).* [15, Theorem 6.1.1]

(i) *The primal problem* (P_u) *is stable with respect to* Φ *if and only if for each solution* x^* *to the primal problem* (P_u) *there exists a solution* V^* *to the dual problem* (D_u) *such that*

$$\Phi\big(x^*, 0\big) \in -\Phi^*\big(0, V^*\big). \tag{2.1}$$

(ii) *Conversely, if* $x^* \in \mathbb{R}^n$ *and* $V^* \in \mathbb{R}^{p\times m}$ *satisfy* (2.1)*, then* x^* *is a solution to* (P_u) *and* V^* *is a solution to* (D_u).

3. Conjugate duality for the constrained vector optimization problem

In this section some special perturbation functions investigated for scalar optimization in [23] are applied to the constrained vector optimization problem. As a consequence, we obtain different dual problems having set-valued objective maps. In analogy to the scalar case, let us call them Lagrange, Fenchel and Fenchel–Lagrange dual problem, respectively. Let $f:\mathbb{R}^n \to \mathbb{R}^p$, $g:\mathbb{R}^n \to \mathbb{R}^m$ be vector-valued functions and $X \subseteq \mathbb{R}^n$. Consider the vector optimization problem

$$(VO) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{ f(x) \,\big|\, x \in G \big\},$$

where $G = \{x \in X \mid g(x) \underset{\mathbb{R}^m_+}{\leqslant} 0\}$. Let us introduce now the following perturbation functions (cf. [5,23]):

$$\Phi_1:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{\infty\}, \quad \Phi_1(x,u) = \begin{cases} f(x), & x \in X,\ g(x) \underset{\mathbb{R}^m_+}{\leqslant} u, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\Phi_2:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \cup \{\infty\}, \quad \Phi_2(x,v) = \begin{cases} f(x+v), & x \in G, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\Phi_3:\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{\infty\}, \quad \Phi_3(x,v,u) = \begin{cases} f(x+v), & x \in X,\ g(x) \underset{\mathbb{R}^m_+}{\leqslant} u, \\ \infty, & \text{otherwise}. \end{cases}$$

Then the corresponding value functions can be written as follows:

$$\begin{aligned} \Psi_1:\mathbb{R}^m \rightrightarrows \mathbb{R}^p, \quad \Psi_1(u) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{ \Phi_1(x,u) \,\big|\, x \in \mathbb{R}^n \big\} \\ &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ f(x) \,\Big|\, x \in X,\ g(x) \underset{\mathbb{R}^m_+}{\leqslant} u \Big\}, \end{aligned}$$

$$\Psi_2:\mathbb{R}^n \rightrightarrows \mathbb{R}^p, \quad \Psi_2(v) = \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{ \Phi_2(x,v) \,\big|\, x \in \mathbb{R}^n \big\} = \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{ f(x+v) \,\big|\, x \in G \big\},$$

$$\begin{aligned} \Psi_3:\mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p, \quad \Psi_3(v,u) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{ \Phi_3(x,v,u) \,\big|\, x \in \mathbb{R}^n \big\} \\ &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ f(x+v) \,\Big|\, x \in X,\ g(x) \underset{\mathbb{R}^m_+}{\leqslant} u \Big\}. \end{aligned}$$

In view of Definition 2.4, the problem (*VO*) is said to be stable with respect to the perturbation function Φ_i, $i = 1, 2, 3,$ if the value function Ψ_i, $i = 1, 2, 3,$ is subdifferentiable at 0.

Definition 3.1. Let $Z \subseteq \mathbb{R}^n$ be a convex set.

(i) The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is said to be convex, if for any $x_1, x_2 \in Z$, $x_1 \neq x_2$, and $\xi \in [0,1]$, we have $\xi G(x_1) + (1-\xi)G(x_2) \subseteq G(\xi x_1 + (1-\xi)x_2) + \mathbb{R}^p_+$.
(ii) The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is said to be strictly convex, if for any $x_1, x_2 \in Z$, $x_1 \neq x_2$, and $\xi \in (0,1)$, we have $\xi G(x_1) + (1-\xi)G(x_2) \subseteq G(\xi x_1 + (1-\xi)x_2) + \operatorname{int}\mathbb{R}^p_+$, where $\operatorname{int}\mathbb{R}^p_+$ denotes the interior of $\mathbb{R}^p_+$.

Lemma 3.1. *Let $X \subseteq \mathbb{R}^n$ be a convex set and f_i, $i = \overline{1,p}$, g_j, $j = \overline{1,m}$, be convex functions. If $\forall u \in \mathbb{R}^m$ (respectively $\forall v \in \mathbb{R}^n$ and $\forall (v,u) \in \mathbb{R}^n \times \mathbb{R}^m$) the set $\Psi_1(u)$ (respectively $\Psi_2(v)$ and $\Psi_3(v,u)$) is externally stable, then the value function Ψ_1 (respectively Ψ_2 and Ψ_3) is convex.*

Proof. Let us verify it only for Ψ_1. In the same way, one can prove the assertions for Ψ_2 and Ψ_3. Let $u_1, u_2 \in \mathbb{R}^m$ and $\lambda \in [0,1]$. Then $\lambda\Psi_1(u_1) + (1-\lambda)\Psi_1(u_2) \subseteq \lambda H_1(u_1) + (1-\lambda)H_1(u_2)$, where H_1 is defined by $H_1(u) := \{f(x) \mid x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u\}$. By the convexity and the external stability, we have

$$\begin{aligned}
&\lambda H_1(u_1) + (1-\lambda)H_1(u_2)\\
&\quad \subseteq \Big\{ f\big(\lambda x + (1-\lambda)z\big) \;\big|\; \lambda x + (1-\lambda)z \in X,\\
&\qquad g\big(\lambda x + (1-\lambda)z\big) \leqq_{\mathbb{R}^m_+} \lambda u_1 + (1-\lambda)u_2 \Big\} + \mathbb{R}^p_+\\
&\quad = H_1\big(\lambda u_1 + (1-\lambda)u_2\big) + \mathbb{R}^p_+ \subseteq \Psi_1\big(\lambda u_1 + (1-\lambda)u_2\big) + \mathbb{R}^p_+.
\end{aligned}$$

Consequently, one has $\lambda\Psi_1(u_1) + (1-\lambda)\Psi_1(u_2) \subseteq \Psi_1(\lambda u_1 + (1-\lambda)u_2) + \mathbb{R}^p_+$. □

Let us give some stability criteria with respect to the above perturbation functions. Similar results can be found in [20].

Proposition 3.1. (See [1].) *Let $X \subseteq \mathbb{R}^n$ be a convex set and g_j, $j = \overline{1,m}$, be convex functions. Assume that the functions f_i, $i = \overline{1,p}$, are strictly convex.*

(i) *If $\forall u \in \mathbb{R}^m$ the set $\Psi_1(u)$ is externally stable and there exists $x_0 \in X$ such that $-g(x_0) \in \operatorname{int}\mathbb{R}^m_+$, then the problem (VO) is stable with respect to Φ_1.*
(ii) *If $\forall v \in \mathbb{R}^n$ the set $\Psi_2(v)$ is externally stable, then the problem (VO) is stable with respect to Φ_2.*
(iii) *If $\forall (v,u) \in \mathbb{R}^n \times \mathbb{R}^m$ the set $\Psi_3(v,u)$ is externally stable and there exists $x_0 \in X$ such that $-g(x_0) \in \operatorname{int}\mathbb{R}^m_+$, then the problem (VO) is stable with respect to Φ_3.*

In Section 5 we consider the vector optimization problem with linear objective function. Since the hypothesis of strictly convexity is not fulfilled, the above stability criteria cannot be applied. Instead of them we will use Proposition 3.2. Let be $A \in \mathbb{R}^{p\times n}$. Consider the vector optimization problem

$$(P_A) \qquad \min_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax \mid x \in G\}.$$

Before giving a stability criterion for (P_A) with respect to Φ_2, let us mention the following trivial properties.

Remark 3.1. Let $h:\mathbb{R}^n \to \mathbb{R}^p$ be a vector-valued function and $Z \subseteq \mathbb{R}^n$. The following assertions are true:

(i) For any $t \in \mathbb{R}^p$ it holds $\{h(x)+t \mid x \in Z\} = \{h(x) \mid x \in Z\} + t$.
(ii) For any set $A \subseteq \mathbb{R}^p$ it holds $\bigcup_{x\in Z}\{A + h(x)\} = A + \bigcup_{x\in Z}\{h(x)\}$.

For the problem (P_A) we can state the following stability criterion.

Proposition 3.2. *Let the set* $\min_{\mathbb{R}^p_+\setminus\{0\}}\{Ax \mid x \in G\}$ *be externally stable. Then the problem* (P_A) *is stable with respect to* Φ_2.

Proof. Let $f(x) = Ax$, $A \in \mathbb{R}^{p\times n}$. Then, in view of Remark 3.1, one has

$$\begin{aligned}
-\Psi_2^*(T) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{v\in\mathbb{R}^n}\Big[\min_{\mathbb{R}^p_+\setminus\{0\}}\{Ax + Av \mid x \in G\} - Tv\Big] \\
&= \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{v\in\mathbb{R}^n}\Big[Av - Tv + \min_{\mathbb{R}^p_+\setminus\{0\}}\{Ax \mid x \in G\}\Big] \\
&= \min_{\mathbb{R}^p_+\setminus\{0\}}\Big[\big\{(A-T)v \bigm| v \in \mathbb{R}^n\big\} + \min_{\mathbb{R}^p_+\setminus\{0\}}\{Ax \mid x \in G\}\Big].
\end{aligned}$$

As the set $\min_{\mathbb{R}^p_+\setminus\{0\}}\{Ax \mid x \in G\}$ is externally stable, for $T = A$ one has (cf. Corollary 2.1)

$$-\Psi_2^*(A) = \min_{\mathbb{R}^p_+\setminus\{0\}}\min_{\mathbb{R}^p_+\setminus\{0\}}\{Ax \mid x \in G\} = \min_{\mathbb{R}^p_+\setminus\{0\}}\{Ax \mid x \in G\}.$$

In other words, $\forall z \in \min_{\mathbb{R}^p_+\setminus\{0\}}\{Ax \mid x \in G\}$, it holds $z \in -\Psi_2^*(A)$. This means that $\partial\Psi_2(0; z) \neq \emptyset$. □

Lagrange duality. In the following we obtain different dual problems by specializing the perturbation approach. First we construct a dual problem to (VO) by using the perturbation function Φ_1. We prove first the following preliminary result.

Proposition 3.3. *Let* $\Lambda \in \mathbb{R}^{p\times m}$. *Then*

(i) $\Phi_1^*(0,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}}\{\{\Lambda u \mid u \in \mathbb{R}^m_+\} + \{\Lambda g(x) - f(x) \mid x \in X\}\}$.
(ii) *If the set* $\max_{\mathbb{R}^p_+\setminus\{0\}}\{\Lambda u \mid u \in \mathbb{R}^m_+\}$ *is externally stable, then it holds*

$$\Phi_1^*(0,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}}\Big\{\max_{\mathbb{R}^p_+\setminus\{0\}}\big\{\Lambda u \bigm| u \in \mathbb{R}^m_+\big\} + \big\{\Lambda g(x) - f(x) \mid x \in X\big\}\Big\}.$$

Proof. (i) Let $\Lambda \in \mathbb{R}^{p\times m}$. Taking into account Remark 3.1,

$$\begin{aligned}
\Phi_1^*(0,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}}\big\{\Lambda u - \Phi_1(x,u) \bigm| x \in \mathbb{R}^n,\ u \in \mathbb{R}^m\big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}}\Big\{\Lambda u - f(x) \Bigm| x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{x\in X}\Big\{\Lambda u - f(x) \Bigm| g(x) \leqq_{\mathbb{R}^m_+} u\Big\}.
\end{aligned}$$

Setting $\bar{u} := u - g(x)$, we have

$$\begin{aligned}\Phi_1^*(0,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \bigl\{\Lambda g(x) - f(x) + \Lambda\bar{u} \bigm| \bar{u}\in\mathbb{R}^m_+\bigr\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \bigl\{\Lambda g(x) - f(x) + \bigl\{\Lambda\bar{u} \bigm| \bar{u}\in\mathbb{R}^m_+\bigr\}\bigr\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigl\{\bigl\{\Lambda u \bigm| u\in\mathbb{R}^m_+\bigr\} + \bigl\{\Lambda g(x) - f(x) \mid x\in X\bigr\}\bigr\}.\end{aligned}$$

(ii) Follows from Lemma 2.2. □

According to Proposition 3.3, we can define the following dual problem to (*VO*):

$$\begin{aligned}\bigl(D_L^{VO}\bigr)\qquad &\max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda\in\mathbb{R}^{p\times m}} \bigl[-\Phi_1^*(0,\Lambda)\bigr] \\ &\quad = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda\in\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \bigl\{\bigl\{-\Lambda u \bigm| u\in\mathbb{R}^m_+\bigr\} + \bigl\{f(x) - \Lambda g(x) \mid x\in X\bigr\}\bigr\}.\end{aligned}$$

This dual problem may be considered as a kind of Lagrange-type dual problem. This interpretation appears evident and natural in the context of the following derivation of the classical Lagrange dual problem to (*VO*) (cf. [15]).

As applications of Theorems 2.1 and 2.2 we get weak and strong duality results for (*VO*) and (D_L^{VO}).

Proposition 3.4 *(Weak duality).*

$$f(x) + \xi \not\leqq_{\mathbb{R}^p_+\setminus\{0\}} 0, \quad \forall x\in G,\ \forall \xi\in\Phi_1^*(0,\Lambda),\ \forall\Lambda\in\mathbb{R}^{p\times m}.$$

Proposition 3.5 *(Strong duality).*

(i) (*VO*) *is stable with respect to* Φ_1 *if and only if for each solution* x^* *to* (*VO*) *there exists a solution* Λ^* *to* (D_L^{VO}) *such that* $f(x^*) \in -\Phi_1^*(0,\Lambda^*)$.
(ii) *Conversely, if* $x^*\in G$ *and* $\Lambda^*\in\mathbb{R}^{p\times m}$ *satisfy* $f(x^*)\in -\Phi_1^*(0,\Lambda^*)$*, then* x^* *is a solution to* (*VO*) *and* Λ^* *is a solution to* (D_L^{VO}).

Under the external stability condition of the set $\max_{\mathbb{R}^p_+\setminus\{0\}}\{\Lambda q \mid q\in\mathbb{R}^m_+\}$, considering as objective of the dual problem the set-valued map in Proposition 3.3(ii), one can obtain similar results.

Before coming to the next perturbation function, let us, as announced, explain how the problem (D_L^{VO}) turns out to be the classical Lagrange dual problem (cf. [15]) under a certain restriction on the feasible set of the dual. To do this, we assume that the feasible set looks like

$$L = \Bigl\{\Lambda\in\mathbb{R}^{p\times m} \Bigm| \Lambda u \geqq_{\mathbb{R}^p_+} 0,\ \forall u\in\mathbb{R}^m_+\Bigr\} = \bigl\{\Lambda\in\mathbb{R}^{p\times m} \bigm| \Lambda\mathbb{R}^m_+ \subseteq \mathbb{R}^p_+\bigr\}.$$

Then we conclude immediately that

$$\min_{\mathbb{R}^p_+\setminus\{0\}} \bigl\{\Lambda u \bigm| u\in\mathbb{R}^m_+\bigr\} = \{0\}, \quad \forall\Lambda\in L. \tag{3.1}$$

Because of $\Lambda \in L$, by using (3.1), from Lemma 2.1(i) follows

$$\begin{aligned}
\Phi_1^*(0, -\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{\big\{-\Lambda u \bigm| u \in \mathbb{R}^m_+\big\} + \big\{-\Lambda g(x) - f(x) \mid x \in X\big\}\big\} \\
&\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{-\Lambda u \bigm| u \in \mathbb{R}^m_+\big\} + \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{-\Lambda g(x) - f(x) \mid x \in X\big\} \\
&= -\min_{\mathbb{R}^p_+\setminus\{0\}} \big\{\Lambda u \bigm| u \in \mathbb{R}^m_+\big\} + \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{-\Lambda g(x) - f(x) \mid x \in X\big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{-\Lambda g(x) - f(x) \bigm| x \in X\big\}.
\end{aligned}$$

Denoting by $\widetilde{\Phi}(\Lambda) := \max_{\mathbb{R}^p_+\setminus\{0\}}\{-\Lambda g(x) - f(x) \mid x \in X\}$, we get the classical Lagrange dual problem to (VO)

$$\big(\widetilde{D}_L^{VO}\big) \quad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda\in L} \big[-\widetilde{\Phi}(\Lambda)\big] = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda\in L} \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{\Lambda g(x) + f(x) \mid x \in X\big\}.$$

Proposition 3.6 *(Weak duality).* [15, Theorem 5.2.4]

$$f(x) + \xi \not\leq_{\mathbb{R}^p_+\setminus\{0\}} 0, \quad \forall x \in G,\ \forall \xi \in \widetilde{\Phi}(\Lambda),\ \forall \Lambda \in L.$$

Proposition 3.7. ([11, Theorem 8.3.3], see also [15, Theorem 5.2.5(i)].) *Let $x^* \in G$, $\Lambda^* \in L$ such that $f(x^*) \in -\widetilde{\Phi}(\Lambda^*)$. Then $f(x^*)$ is simultaneously a minimal point to the primal problem (VO) and a maximal point to the dual problem $(\widetilde{D}_L^{VO})$.*

Fenchel duality. The following result is in connection with the perturbation function Φ_2.

Proposition 3.8. *Let $T \in \mathbb{R}^{p\times n}$. Then*

(i) $\Phi_2^*(0, T) = \max_{\mathbb{R}^p_+\setminus\{0\}}\{\{Tv - f(v) \mid v \in \mathbb{R}^n\} + \{-Tx \mid x \in G\}\}$.
(ii) *If the set $f^*(T) = \max_{\mathbb{R}^p_+\setminus\{0\}}\{Tv - f(v) \mid v \in \mathbb{R}^n\}$ is externally stable, then it holds*

$$\Phi_2^*(0, T) = \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{f^*(T) + \{-Tx \mid x \in G\}\big\}.$$

Proof. (i) Let $T \in \mathbb{R}^{p\times n}$. In view of Remark 3.1,

$$\begin{aligned}
\Phi_2^*(0, T) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{Tv - \Phi_2(x, v) \bigm| x \in \mathbb{R}^n,\ v \in \mathbb{R}^n\big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{Tv - f(x+v) \bigm| x \in G,\ v \in \mathbb{R}^n\big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in G} \big\{Tv - f(x+v) \bigm| v \in \mathbb{R}^n\big\}.
\end{aligned}$$

Denoting $\bar{v} := x + v$, one gets

$$\Phi_2^*(0, T) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in G} \big\{T\bar{v} - f(\bar{v}) - Tx \bigm| \bar{v} \in \mathbb{R}^n\big\}$$

$$= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in G} \bigl\{-Tx + \bigl\{T\bar{v} - f(\bar{v}) \bigm| \bar{v} \in \mathbb{R}^n\bigr\}\bigr\}$$
$$= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigl\{\bigl\{Tv - f(v) \bigm| v \in \mathbb{R}^n\bigr\} + \{-Tx \mid x \in G\}\bigr\}.$$

(ii) Follows from Lemma 2.2. □

As a consequence we state the following dual problem to (*VO*), which will be called the Fenchel dual problem

$$\bigl(D_F^{VO}\bigr) \quad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \bigl[-\Phi_2^*(0,T)\bigr]$$
$$= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \min_{\mathbb{R}^p_+\setminus\{0\}} \bigl\{\bigl\{f(v) - Tv \bigm| v \in \mathbb{R}^n\bigr\} + \{Tx \mid x \in G\}\bigr\}.$$

Also in this case one can state weak and strong duality assertions.

Proposition 3.9 *(Weak duality).*

$$f(x) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleq} 0, \quad \forall x \in G,\ \forall \xi \in \Phi_2^*(0,T),\ \forall T \in \mathbb{R}^{p\times n}.$$

Proposition 3.10 *(Strong duality).*

(i) (*VO*) *is stable with respect to* Φ_2 *if and only if for each solution* x^* *to* (*VO*), *there exists a solution* T^* *to* (D_F^{VO}) *such that* $f(x^*) \in -\Phi_2^*(0,T^*)$.
(ii) *Conversely, if* $x^* \in G$ *and* $T^* \in \mathbb{R}^{p\times n}$ *satisfy* $f(x^*) \in -\Phi_2^*(0,T^*)$*, then* x^* *is a solution to* (*VO*) *and* T^* *is a solution to* (D_F^{VO}).

As mentioned before, under the external stability of the set $f^*(T) = \max_{\mathbb{R}^p_+\setminus\{0\}}\{Tv - f(v) \mid v \in \mathbb{R}^n\}$, for the dual problem having as objective the set-valued map in Proposition 3.8(ii), one can also show similar dual assertions.

Fenchel–Lagrange duality. In the following we deal with the perturbation function Φ_3.

Proposition 3.11. *Let* $\Lambda \in \mathbb{R}^{p\times m}$ *and* $T \in \mathbb{R}^{p\times n}$. *Then*

(i) $\Phi_3^*(0,T,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}}\{\bigcup_{u\in\mathbb{R}^m_+}\{\Lambda u\} + \bigcup_{v\in\mathbb{R}^n}\{Tv - f(v)\} + \bigcup_{x\in X}\{\Lambda g(x) - Tx\}\}$.
(ii) *If the sets* $\max_{\mathbb{R}^p_+\setminus\{0\}}\{\Lambda u \mid u \in \mathbb{R}^m_+\}$ *and* $f^*(T)$ *are externally stable, then it holds*

$$\Phi_3^*(0,T,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{u\in\mathbb{R}^m_+} \{\Lambda u\} + f^*(T) + \bigcup_{x\in X} \bigl\{\Lambda g(x) - Tx\bigr\} \right\}.$$

Proof. (i) Let $T \in \mathbb{R}^{p\times n}$ and $\Lambda \in \mathbb{R}^{p\times m}$. By applying Remark 3.1,

$$\Phi_3^*(0,T,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigl\{Tv + \Lambda u - \Phi_3(x,v,u) \bigm| x \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\ u \in \mathbb{R}^m\bigr\}$$
$$= \max_{\mathbb{R}^p_+\setminus\{0\}} \Bigl\{Tv + \Lambda u - f(x+v) \Bigm| x \in X,\ v \in \mathbb{R}^n,\ g(x) \underset{\mathbb{R}^m_+}{\leqq} u\Bigr\}$$

$$= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\substack{x\in X\\ v\in\mathbb{R}^n}} \Big\{Tv + \Lambda u - f(x+v) \;\Big|\; g(x) \leqq_{\mathbb{R}^m_+} u\Big\}.$$

Putting $\bar{u} := u - g(x)$, one has

$$\begin{aligned}
\Phi_3^*(0,T,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\substack{x\in X\\ v\in\mathbb{R}^n}} \big\{Tv + \Lambda\big(g(x)+\bar{u}\big) - f(x+v) \;\big|\; \bar{u}\in\mathbb{R}^m_+\big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \big\{Tv + \Lambda g(x) - f(x+v) + \big\{\Lambda\bar{u} \;\big|\; \bar{u}\in\mathbb{R}^m_+\big\}\big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \big\{\Lambda g(x) + \big\{\Lambda u \;\big|\; u\in\mathbb{R}^m_+\big\} + \big\{Tv - f(x+v) \;\big|\; v\in\mathbb{R}^n\big\}\big\}.
\end{aligned}$$

Setting $\bar{v} := x + v$, we obtain

$$\begin{aligned}
\Phi_3^*(0,T,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \big\{\Lambda g(x) + \big\{\Lambda u \;\big|\; u\in\mathbb{R}^m_+\big\} + \big\{T\bar{v} - Tx - f(\bar{v}) \;\big|\; \bar{v}\in\mathbb{R}^n\big\}\big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \big\{\Lambda g(x) - Tx + \big\{\Lambda u \;\big|\; u\in\mathbb{R}^m_+\big\} + \big\{T\bar{v} - f(\bar{v}) \;\big|\; \bar{v}\in\mathbb{R}^n\big\}\big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{\big\{\Lambda u \;\big|\; u\in\mathbb{R}^m_+\big\} + \big\{Tv - f(v) \;\big|\; v\in\mathbb{R}^n\big\} + \big\{\Lambda g(x) - Tx \;\big|\; x\in X\big\}\big\}.
\end{aligned}$$

(ii) By Lemma 2.2, one can easily verify (ii). □

Now we can formulate the following so-called Fenchel–Lagrange dual problem to (VO)

$$\begin{aligned}
\big(D^{VO}_{FL}\big) \quad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \big[-\Phi_3^*(0,T,\Lambda)\big]\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{\big\{f(v) - Tv \;\big|\; v\in\mathbb{R}^n\big\}\\
&\quad + \big\{-\Lambda u \;\big|\; u\in\mathbb{R}^m_+\big\} + \big\{Tx - \Lambda g(x) \;\big|\; x\in X\big\}\big\}.
\end{aligned}$$

Proposition 3.12 *(Weak duality).*

$$f(x) + \xi \not\leq_{\mathbb{R}^p_+\setminus\{0\}} 0, \quad \forall x\in X,\ \forall\xi\in\Phi_3^*(0,T,\Lambda),\ \forall(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}.$$

Proposition 3.13 *(Strong duality).*

(i) (VO) *is stable with respect to* Φ_3 *if and only if for each solution* x^* *to* (VO) *there exists a solution* (T^*,Λ^*) *to* (D^{VO}_{FL}) *such that* $f(x^*)\in-\Phi_3^*(0,T^*,\Lambda^*)$.
(ii) *Conversely, if* $x^*\in X$ *and* $(T^*,\Lambda^*)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}$ *satisfy* $f(x^*)\in-\Phi_3^*(0,T^*,\Lambda^*)$*, then* x^* *is a solution to* (VO) *and* (T^*,Λ^*) *is a solution to* (D^{VO}_{FL}).

Like for $(\widetilde{D}^{VO}_L)$, under the same restriction on Λ, we can introduce another Fenchel–Lagrange-type dual problem. Indeed, let us suppose that $\Lambda\in L$. Then, according to Lemma 2.1(i) and (3.1), it holds

$$\begin{aligned}
\Phi_3^*(0,T,-\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}}\left\{\bigcup_{u\in\mathbb{R}^m_+}\{-\Lambda u\}+\bigcup_{v\in\mathbb{R}^n}\big\{Tv-f(v)\big\}+\bigcup_{x\in X}\big\{-\Lambda g(x)-Tx\big\}\right\}\\
&\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{u\in\mathbb{R}^m_+}\{-\Lambda u\}+\max_{\mathbb{R}^p_+\setminus\{0\}}\left\{\bigcup_{v\in\mathbb{R}^n}\big\{Tv-f(v)\big\}+\bigcup_{x\in X}\big\{-\Lambda g(x)-Tx\big\}\right\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}}\left\{\bigcup_{v\in\mathbb{R}^n}\big\{Tv-f(v)\big\}+\bigcup_{x\in X}\big\{-\Lambda g(x)-Tx\big\}\right\}.
\end{aligned}$$

Let us denote by $\widetilde{\Psi}(T,\Lambda) := \max_{\mathbb{R}^p_+\setminus\{0\}}\{\bigcup_{v\in\mathbb{R}^n}\{Tv-f(v)\}+\bigcup_{x\in X}\{-\Lambda g(x)-Tx\}\}$. If the set $f^*(T)$ is externally stable, then $\widetilde{\Psi}(T,\Lambda)$ can be rewritten as $\widetilde{\Psi}(T,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}}\{f^*(T)+\bigcup_{x\in X}\{-\Lambda g(x)-Tx\}\}$. So we can suggest the following dual problem

$$\begin{aligned}
\big(\widetilde{D}^{VO}_{FL}\big)\quad &\max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times L}\big[-\widetilde{\Psi}(T,\Lambda)\big]\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times L}\min_{\mathbb{R}^p_+\setminus\{0\}}\left\{\bigcup_{v\in\mathbb{R}^n}\big\{f(v)-Tv\big\}+\bigcup_{x\in X}\big\{Tx+\Lambda g(x)\big\}\right\}.
\end{aligned}$$

Proposition 3.14 *(Weak duality).*

$$f(x)+\xi \nleqslant_{\mathbb{R}^p_+\setminus\{0\}} 0,\quad \forall x\in G,\ \forall\xi\in\widetilde{\Psi}(T,\Lambda),\ \forall(T,\Lambda)\in\mathbb{R}^{p\times n}\times L.$$

Proof. Let $(T,\Lambda)\in\mathbb{R}^{p\times n}\times L$ be fixed and $\xi\in\widetilde{\Psi}(T,\Lambda)$. In other words

$$\xi \nleqslant_{\mathbb{R}^p_+\setminus\{0\}} Tv-f(v)+\big(-\Lambda g(x)-Tx\big),\quad \forall v\in\mathbb{R}^n,\ \forall x\in X.$$

Choosing $v=x:=\bar{x}\in G$, we obtain that $f(\bar{x})+\xi \nleqslant_{\mathbb{R}^p_+\setminus\{0\}} -\Lambda g(\bar{x})$. On the other hand, since $\Lambda\in L$ from $\bar{x}\in G$ it follows that $-\Lambda g(\bar{x}) \geqslant_{\mathbb{R}^p_+} 0$. Consequently, one has $f(x)+\xi \nleqslant_{\mathbb{R}^p_+\setminus\{0\}} 0$. □

Proposition 3.15. *Let $x^*\in G$ and $(T^*,\Lambda^*)\in\mathbb{R}^{p\times n}\times L$ be such that $f(x^*)\in-\widetilde{\Psi}(T^*,\Lambda^*)$. Then $f(x^*)$ is simultaneously a minimal point of the primal problem (VO) and a maximal point to the dual problem $(\widetilde{D}^{VO}_{FL})$.*

Proof. Let $x^*\in G$ and $(T^*,\Lambda^*)\in\mathbb{R}^{p\times n}\times L$ be such that $f(x^*)\in-\widetilde{\Psi}(T^*,\Lambda^*)$. The latter means

$$f\big(x^*\big)\in\min_{\mathbb{R}^p_+\setminus\{0\}}\left\{\bigcup_{v\in\mathbb{R}^n}\big\{f(v)-T^*v\big\}+\bigcup_{x\in X}\big\{T^*x+\Lambda^*g(x)\big\}\right\}.\tag{3.2}$$

If $f(x^*)$ is not a minimal point of the primal problem (*VO*), then there exists $x\in G$ such that $f(x) \leqslant_{\mathbb{R}^p_+\setminus\{0\}} f(x^*)$. As mentioned before, since $\Lambda^*\in L$, $x\in G$ yields $\Lambda^*g(x)\leqslant_{\mathbb{R}^p_+} 0$.

Consequently, we have $f(x) + \Lambda^* g(x) \underset{\mathbb{R}^p_+\setminus\{0\}}{\leqslant} f(x^*)$ or, equivalently, $f(x) - T^*x + T^*x + \Lambda^* g(x) \underset{\mathbb{R}^p_+\setminus\{0\}}{\leqslant} f(x^*)$. But

$$f(x) - T^*x + T^*x + \Lambda^* g(x) \in \bigcup_{v\in\mathbb{R}^n} \{f(v) - T^*v\} + \bigcup_{x\in X} \{T^*x + \Lambda^* g(x)\},$$

which is a contradiction to (3.2). Therefore $f(x^*)$ is a minimal point to the problem (*VO*). Further, if $f(x^*)$ is not a solution to $(\widetilde{D}^{VO}_{FL})$, then $\exists \tilde{y} \in \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times L}[-\widetilde{\Psi}(T,\Lambda)]$ such that $f(x^*) \underset{\mathbb{R}^p_+\setminus\{0\}}{\leqslant} \tilde{y}$. Let $(\widetilde{T}, \widetilde{\Lambda}) \in \mathbb{R}^{p\times n} \times L$ be such that $\tilde{y} \in -\widetilde{\Psi}(\widetilde{T}, \widetilde{\Lambda})$. From $\widetilde{\Lambda} g(x^*) \underset{\mathbb{R}^p_+}{\leqslant} 0$ follows

$$\tilde{y} \underset{\mathbb{R}^p_+\setminus\{0\}}{\geqslant} f(x^*) + \widetilde{\Lambda} g(x^*) = f(x^*) - \widetilde{T}x^* + \widetilde{T}x^* + \widetilde{\Lambda} g(x^*),$$

which contradicts the fact that $\tilde{y} \in -\widetilde{\Psi}(\widetilde{T}, \widetilde{\Lambda})$ in the same way as before. Accordingly, $f(x^*)$ is a solution of $(\widetilde{D}^{VO}_{FL})$. □

4. Special cases

This section aims to investigate some special cases of dual problems based on alternative definitions of the conjugate maps and the subgradient for a set-valued map having vector variables. In Definition 2.3, if we choose $U := [t, \dots, t]^T \in \mathbb{R}^{p\times n}$ for $t \in \mathbb{R}^n$, as variable of the conjugate maps, then this reduces to the definition considered in this section. Remark that duality results for vector optimization developed by Tanino and Sawaragi (see [15,20]) are essentially not distinguishable in both cases. The advantage of considering conjugate maps with vector variable consists in the fact that the corresponding dual problems have a more simple form than ones in Section 3 and so they can be easily reduced to the duals in scalar optimization. Let us recall first the definitions of the conjugate maps with vector variables (cf. Definition 2.3).

Definition 4.1 *(The type II Fenchel transform).* [11, Definition 7.2.3] Let $h : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map. For $\lambda, x \in \mathbb{R}^n$ we denote $(\lambda^T x)_p = (\lambda^T x, \dots, \lambda^T x)^T \in \mathbb{R}^p$.

(i) The set-valued map $h^*_p : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ defined by $h^*_p(\lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} [(\lambda^T x)_p - h(x)]$, $\lambda \in \mathbb{R}^n$ is called the (type II) conjugate map of h;

(ii) the conjugate map of h^*_p, $h^{**}_p(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda\in\mathbb{R}^n} [(\lambda^T x)_p - h^*_p(\lambda)]$, $x \in \mathbb{R}^n$ is called the biconjugate map of h;

(iii) $\lambda \in \mathbb{R}^n$ is said to be a subgradient of the set-valued map h at $(\bar{x}; \bar{y})$, if $\bar{y} \in h(\bar{x})$ and $\bar{y} - (\lambda^T \bar{x})_p \in \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} [h(x) - (\lambda^T x)_p]$.

Like in Section 3, let $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^n \to \mathbb{R}^m$ be vector-valued functions and $X \subseteq \mathbb{R}^n$. Based on the perturbation functions introduced in Section 3, let us suggest some dual problems having now vector variables. For convenience, in this section we denote the perturbation functions by φ_i instead of Φ_i, for $i = 1, 2, 3$. Let us notice that throughout this section instead of φ^*_{ip} we write φ^*_i, $i = 1, 2, 3$.

Lagrange duality. By using the dual objective map having a vector variable with respect to φ_1, the Lagrange dual problem to (*VO*) was introduced in [20]. Let us now explain how one can obtain this dual.

Lemma 4.1. *Let* $\lambda \in \mathbb{R}^m$. *Then* $\min_{\mathbb{R}^p_+ \setminus \{0\}}\{(\lambda^T x)_p \mid x \in \mathbb{R}^m_+\} = \{0\}$, *if* $\lambda \underset{\mathbb{R}^m_+}{\geqslant} 0$ *and is equal* $\emptyset$, *otherwise.*

Proof. Let $z \in \min_{\mathbb{R}^p_+ \setminus \{0\}}\{(\lambda^T x)_p \mid x \in \mathbb{R}^m_+\}$. Then $\exists \bar{x} \in \mathbb{R}^m_+$ such that $z = (\lambda^T \bar{x})_p$ and it holds $(\lambda^T \bar{x})_p \underset{\mathbb{R}^p_+ \setminus \{0\}}{\ngeqslant} (\lambda^T x)_p$, $\forall x \in \mathbb{R}^m_+$ or, equivalently, $\lambda^T \bar{x} \leqslant \lambda^T x$, $\forall x \in \mathbb{R}^m_+$. In other words, it holds $\lambda^T \bar{x} = \min_{x \in \mathbb{R}^n} \lambda^T x$. Since $\inf_{x \in \mathbb{R}^m_+} \lambda^T x = 0$, if $\lambda \underset{\mathbb{R}^m_+}{\geqslant} 0$, and is equal $-\infty$, otherwise, we obtain the conclusion. $\square$

Proposition 4.1. *Let* $\lambda \in \mathbb{R}^m$. *Then* $\varphi_1^*(0, \lambda) = \max_{\mathbb{R}^p_+ \setminus \{0\}}\{(\lambda^T g(x))_p - f(x) \mid x \in X\}$, *if* $\lambda \underset{\mathbb{R}^m_+}{\leqslant} 0$ *and is equal* $\emptyset$, *otherwise.*

Proof. Let $\lambda \in \mathbb{R}^m$. Then by definition

$$\begin{aligned}
\varphi_1^*(0, \lambda) &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(\lambda^T u\right)_p - \varphi_1(x, u) \,\middle|\, x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \right\} \\
&= \max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(\lambda^T u\right)_p - f(x) \,\middle|\, x \in X,\ g(x) \underset{\mathbb{R}^m_+}{\leqslant} u \right\}.
\end{aligned}$$

Setting $\bar{u} := u - g(x)$, we have

$$\begin{aligned}
\varphi_1^*(0, \lambda) &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(\lambda^T g(x)\right)_p + \left(\lambda^T \bar{u}\right)_p - f(x) \,\middle|\, x \in X,\ \bar{u} \in \mathbb{R}^m_+ \right\} \\
&= \max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left\{ \left(\lambda^T g(x)\right)_p - f(x) \,\middle|\, x \in X \right\} + \left\{ \left(\lambda^T \bar{u}\right)_p \mid \bar{u} \in \mathbb{R}^m_+ \right\} \right\}.
\end{aligned}$$

In view of Lemmas 2.1(i) and 4.1, one has

$$\begin{aligned}
\varphi_1^*(0, \lambda) &\subseteq \max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(\lambda^T g(x)\right)_p - f(x) \,\middle|\, x \in X \right\} + \max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(\lambda^T u\right)_p \,\middle|\, u \in \mathbb{R}^m_+ \right\} \\
&= \max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(\lambda^T g(x)\right)_p - f(x) \,\middle|\, x \in X \right\} - \min_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(-\lambda^T u\right)_p \,\middle|\, u \in \mathbb{R}^m_+ \right\}
\end{aligned}$$

being equal $\max_{\mathbb{R}^p_+ \setminus \{0\}}\{(\lambda^T g(x))_p - f(x) \mid x \in X\}$, if $\lambda \underset{\mathbb{R}^m_+}{\leqslant} 0$ and equal $\emptyset$, otherwise. For $\lambda \underset{\mathbb{R}^m_+}{\leqslant} 0$ it remains to show that $\max_{\mathbb{R}^p_+ \setminus \{0\}}\{(\lambda^T g(x))_p - f(x) \mid x \in X\} \subseteq \varphi_1^*(0, \lambda)$. Let $\bar{y} \in \max_{\mathbb{R}^p_+ \setminus \{0\}}\{(\lambda^T g(x))_p - f(x) \mid x \in X\}$. This means $\bar{y} \in \{(\lambda^T g(x))_p - f(x) \mid x \in X\}$ and $\bar{y} \underset{\mathbb{R}^p_+ \setminus \{0\}}{\nleqslant} (\lambda^T g(x))_p - f(x)$, $\forall x \in X$. Choosing $\bar{u} = 0$, we have

$$\bar{y} = \bar{y} + \left(\lambda^T \bar{u}\right)_p \in \left\{ \left\{ \left(\lambda^T g(x)\right)_p - f(x) \,\middle|\, x \in X \right\} + \left\{ \left(\lambda^T u\right)_p \,\middle|\, u \in \mathbb{R}^m_+ \right\} \right\}.$$

On the other hand, since $(\lambda^T u)_p \underset{\mathbb{R}^p_+}{\leqslant} 0$, $\forall u \in \mathbb{R}^m_+$, one has $\bar{y} \underset{\mathbb{R}^p_+}{\geqslant} \bar{y} + (\lambda^T u)_p$ and it holds

$$\bar{y} + \left(\lambda^T u\right)_p \underset{\mathbb{R}^p_+ \setminus \{0\}}{\nleqslant} \left(\lambda^T g(x)\right)_p - f(x) + \left(\lambda^T u\right)_p, \quad \forall x \in X,\ \forall u \in \mathbb{R}^m_+.$$

Consequently, we obtain that $\bar{y} \underset{\mathbb{R}^p_+\backslash\{0\}}{\nleqslant} (\lambda^T g(x))_p - f(x) + (\lambda^T u)_p,\ \forall x \in X,\ \forall u \in \mathbb{R}^m_+$. In other words $\bar{y} \in \varphi_1^*(0,\lambda)$. $\square$

Thus a dual problem to (VO) can be formulated as

$$(\widehat{D}_L^{VO}) \quad \max_{\mathbb{R}^p_+\backslash\{0\}} \bigcup_{\lambda\in\mathbb{R}^m} \left[-\varphi_1^*(0,\lambda)\right] = \max_{\mathbb{R}^p_+\backslash\{0\}} \bigcup_{\lambda \leqslant_{\mathbb{R}^m_+} 0} \min_{\mathbb{R}^p_+\backslash\{0\}} \left\{ f(x) - \left(\lambda^T g(x)\right)_p \,\middle|\, x \in X \right\}$$

$$= \max_{\mathbb{R}^p_+\backslash\{0\}} \bigcup_{\lambda \underset{\mathbb{R}^m_+}{\geqslant} 0} \min_{\mathbb{R}^p_+\backslash\{0\}} \left\{ f(x) + \left(\lambda^T g(x)\right)_p \,\middle|\, x \in X \right\}.$$

Proposition 4.2. [15, Theorem 6.1.4]

(i) *The problem (VO) is stable with respect to φ_1 if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geqslant} 0$ to the dual problem $(\widehat{D}_L^{VO})$ such that $f(\bar{x}) \in \min_{\mathbb{R}^p_+\backslash\{0\}}\{f(x) + (\bar{\lambda}^T g(x))_p \mid x \in X\}$. In this case $\bar{\lambda}^T g(\bar{x}) = 0$.*

(ii) *Conversely, if $\bar{x} \in G$ and $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geqslant} 0$ satisfy the relations above, then $\bar{x}$ and $\bar{\lambda}$ are solutions to (VO) and $(\widehat{D}_L^{VO})$, respectively.*

Remark 4.1. Let $p = 1$ and the assumptions of Theorem 2.8 in [5] (see also [23]) be fulfilled. Then Proposition 4.2 coincides with the optimality conditions (cf. [5, Theorem 2.9]) for the Lagrange dual problem in scalar optimization.

Example 4.1. Consider the vector optimization problem

$$(VO_1) \quad \min_{\mathbb{R}^2_+\backslash\{0\}} \left\{ (x_1, x_2) \,\middle|\, 0 \leqslant x_i \leqslant 1,\ x_i \in \mathbb{R},\ i = 1, 2 \right\}.$$

Let us construct the Lagrange dual problem to (VO_1). Before doing this, in view of $(\widehat{D}_L^{VO})$, for $\lambda \underset{\mathbb{R}^m_+}{\geqslant} 0$, one has to calculate $\min_{\mathbb{R}^p_+\backslash\{0\}}\{f(x) + (\lambda^T g(x))_p \mid x \in X\}$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T \in \mathbb{R}^4$ and the function $g : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $g(x) = (-x_1, x_1 - 1, -x_2, x_2 - 1)^T$. In other words, we have

$$\min_{\mathbb{R}^2_+\backslash\{0\}} \left\{ \begin{pmatrix} (\lambda_2 - \lambda_1 + 1)x_1 + (\lambda_4 - \lambda_3)x_2 \\ (\lambda_2 - \lambda_1)x_1 + (\lambda_4 - \lambda_3 + 1)x_2 \end{pmatrix} \,\middle|\, (x_1, x_2)^T \in \mathbb{R}^2 \right\} - \begin{pmatrix} \lambda_2 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix}.$$

Let $B_1 = \begin{pmatrix} \lambda_2-\lambda_1+1 & \lambda_4-\lambda_3 \\ \lambda_2-\lambda_1 & \lambda_4-\lambda_3+1 \end{pmatrix}$. Taking into account Theorem 11.20 in [12], if $\exists \mu \in \operatorname{int} \mathbb{R}^2_+$ such that $\mu^T B_1 = 0$, then $\min_{\mathbb{R}^2_+\backslash\{0\}}\{B_1 x \mid x \in \mathbb{R}^2\} = \{B_1 x \mid x \in \mathbb{R}^2\}$ (cf. Lemma 5.1). In the other case, one has $\min_{\mathbb{R}^2_+\backslash\{0\}}\{B_1 x \mid x \in \mathbb{R}^2\} = \emptyset$. As $\mu^T B_1 = 0$ is nothing else than $(\lambda_2 - \lambda_1 + 1)\mu_1 + (\lambda_2 - \lambda_1)\mu_2 = 0$ and $(\lambda_4 - \lambda_3)\mu_1 + (\lambda_4 - \lambda_3 + 1)\mu_2 = 0$, it must hold $\lambda_1 = \lambda_2 + \frac{\mu_1}{\mu_1+\mu_2}$ and $\lambda_3 = \lambda_4 + \frac{\mu_2}{\mu_1+\mu_2}$. Now let us define

$$L_1 := \left\{ \lambda \in \mathbb{R}^4 \,\middle|\, \exists \mu \in \operatorname{int} \mathbb{R}^2_+ \text{ such that } \lambda_1 = \lambda_2 + \frac{\mu_1}{\mu_1 + \mu_2},\ \lambda_3 = \lambda_4 + \frac{\mu_2}{\mu_1 + \mu_2} \right\}.$$

In conclusion, we obtain for the Lagrange dual problem $(\widehat{D}_L^{VO_1})$ the following formulation

$$\max_{\mathbb{R}^2_+\setminus\{0\}} \bigcup_{\substack{\lambda \underset{\mathbb{R}^4_+}{\geqslant} 0 \\ \lambda\in L_1}} \left\{ \begin{pmatrix} (\lambda_2-\lambda_1+1)x_1+(\lambda_4-\lambda_3)x_2 \\ (\lambda_2-\lambda_1)x_1+(\lambda_4-\lambda_3+1)x_2 \end{pmatrix} - \begin{pmatrix} \lambda_2+\lambda_4 \\ \lambda_2+\lambda_4 \end{pmatrix} \,\middle|\, (x_1,x_2)^T \in \mathbb{R}^2 \right\}.$$

Let $\bar{x} = (0,0)^T \in \mathbb{R}^2$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4)^T \in L_1$ be such that $\bar{\lambda} \underset{\mathbb{R}^4_+}{\geqslant} 0$ and $\bar{\lambda}^T g(\bar{x}) = 0$. Then from $\bar{\lambda}^T g(\bar{x}) = 0$ follows $\bar{\lambda}_2 + \bar{\lambda}_4 = 0$. As $\bar{\lambda}_2, \bar{\lambda}_4 \geqslant 0$, this implies that $\bar{\lambda}_2 = \bar{\lambda}_4 = 0$. Moreover, as $\bar{\lambda} \in L_1$, it holds $\bar{\lambda}_1 = \frac{\mu_1}{\mu_1+\mu_2}$ and $\bar{\lambda}_3 = \frac{\mu_2}{\mu_1+\mu_2}$. In other words, $\bar{\lambda}_1 = \alpha := \frac{\mu_1}{\mu_1+\mu_2}$, $\bar{\lambda}_3 = 1-\alpha$, for $0 < \alpha < 1$. On the other hand, it is clear that

$$\begin{aligned} f(\bar{x}) = (0,0)^T &\in \min_{\mathbb{R}^2_+\setminus\{0\}} \left\{ f(x) + \left(\bar{\lambda}^T g(x)\right)_2 \,\middle|\, x \in \mathbb{R}^2 \right\} \\ &= \left\{ \begin{pmatrix} \frac{\mu_2}{\mu_1+\mu_2}(x_1-x_2) \\ \frac{\mu_1}{\mu_1+\mu_2}(x_2-x_1) \end{pmatrix} \,\middle|\, (x_1,x_2)^T \in \mathbb{R}^2 \right\} \\ &= \left\{ \begin{pmatrix} (\alpha-1)y \\ \alpha y \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}, \quad 0 < \alpha < 1. \end{aligned}$$

According to Proposition 4.2(ii), $\bar{x} = (0,0)^T$ and $\bar{\lambda} = (\alpha, 0, 1-\alpha, 0)^T$, $0 < \alpha < 1$ are solutions to (VO_1) and $(\widehat{D}_L^{VO_1})$, respectively.

Fenchel duality. Before considering the next dual problem, we prove the following assertions.

Lemma 4.2. *Let $t \in \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$. If the set $\max_{\mathbb{R}^p_+\setminus\{0\}}\{(t^T x)_p \mid x \in Y\}$ is not empty, then we have $\max_{\mathbb{R}^p_+\setminus\{0\}}\{(t^T x)_p \mid x \in Y\} = \{(\max_{x\in Y} t^T x)_p\}$.*

Proof. Let $t \in \mathbb{R}^n$. By assumption, there exists $\bar{x} \in Y$ such that $(t^T \bar{x})_p \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleq} (t^T x)_p$, $\forall x \in Y$ or, equivalently, $t^T \bar{x} \geqslant t^T x$, $\forall x \in Y$. Therefore $t^T \bar{x} = \max_{x\in Y} t^T x$. $\square$

Proposition 4.3. *Let $t \in \mathbb{R}^n$. Then $\varphi_2^*(0,t) = f_p^*(t) - (\min_{x\in G} t^T x)_p$, if $\max_{\mathbb{R}^p_+\setminus\{0\}}\{(-t^T x)_p \mid x \in G\} \neq \emptyset$ and is equal $\emptyset$, otherwise.*

Proof. Let $t \in \mathbb{R}^n$. By definition

$$\begin{aligned} \varphi_2^*(0,t) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(t^T v\right)_p - \varphi_2(x,v) \,\middle|\, x \in \mathbb{R}^n,\ v \in \mathbb{R}^n \right\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(t^T v\right)_p - f(x+v) \,\middle|\, x \in G,\ v \in \mathbb{R}^n \right\}. \end{aligned}$$

Substituting $\bar{v} := x + v$, we get

$$\begin{aligned} \varphi_2^*(0,t) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(t^T \bar{v}\right)_p - (t^T x)_p - f(\bar{v}) \,\middle|\, x \in G,\ \bar{v} \in \mathbb{R}^n \right\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left\{ \left(t^T \bar{v}\right)_p - f(\bar{v}) \,\middle|\, \bar{v} \in \mathbb{R}^n \right\} + \left\{ \left(-t^T x\right)_p \,\middle|\, x \in G \right\} \right\}. \end{aligned}$$

According to Lemma 2.1(i), it follows that

$$\varphi_2^*(0,t) \subseteq \max_{\mathbb{R}_+^p\setminus\{0\}} \left\{ \left(t^T v\right)_p - f(v) \,\middle|\, v \in \mathbb{R}^n \right\} + \max_{\mathbb{R}_+^p\setminus\{0\}} \left\{ \left(-t^T x\right)_p \,\middle|\, x \in G \right\}.$$

It is clear that unless $\max_{\mathbb{R}_+^p\setminus\{0\}}\{(-t^T x)_p \mid x \in G\} \neq \emptyset$, $\varphi_2^*(0,t) = \emptyset$.

Since $\max_{\mathbb{R}_+^p\setminus\{0\}}\{(-t^T x)_p \mid x \in G\} \neq \emptyset$, by Lemma 4.2 it holds $\max_{\mathbb{R}_+^p\setminus\{0\}}\{(-t^T x)_p \mid x \in G\} = \{(-\min_{x\in G} t^T x)_p\}$. In other words

$$\varphi_2^*(0,t) \subseteq \max_{\mathbb{R}_+^p\setminus\{0\}} \left\{ \left(t^T v\right)_p - f(v) \,\middle|\, v \in \mathbb{R}^n \right\} - \left(\min_{x\in G} t^T x\right)_p = f_p^*(t) - \left(\min_{x\in G} t^T x\right)_p.$$

Let now $\bar{y} \in f_p^*(t) - (\min_{x\in G} t^T x)_p$. Then $\bar{y} \in \max_{\mathbb{R}_+^p\setminus\{0\}}\{\{(t^T v)_p - f(v) \mid v \in \mathbb{R}^n\} - (\min_{x\in G} t^T x)_p\}$. This means that $\bar{y} \underset{\mathbb{R}_+^p\setminus\{0\}}{\nleqslant} (t^T v)_p - f(v) - (\min_{x\in G} t^T x)_p$, $\forall v \in \mathbb{R}^n$. Moreover, from

$$\left(t^T v\right)_p - f(v) - \left(\min_{x\in G} t^T x\right)_p \underset{\mathbb{R}_+^p}{\geqslant} \left(t^T v\right)_p - f(v) - \left(t^T x\right)_p, \quad \forall x \in G,\ \forall v \in \mathbb{R}^n,$$

follows $(t^T v)_p - f(v) - (t^T x)_p \underset{\mathbb{R}_+^p\setminus\{0\}}{\ngeqslant} \bar{y}$, $\forall x \in G$, $\forall v \in \mathbb{R}^n$. Whence $\bar{y} \in \varphi_2^*(0,t)$. □

The Fenchel dual problem can be stated now as being

$$\left(\widehat{D}_F^{VO}\right) \quad \max_{\mathbb{R}_+^p\setminus\{0\}} \bigcup_{t\in\mathbb{R}^n} \left[-\varphi_2^*(0,t)\right] = \max_{\mathbb{R}_+^p\setminus\{0\}} \bigcup_{t\in\mathbb{R}^n} \left[-f_p^*(t) + \left(\min_{x\in G} t^T x\right)_p\right].$$

From Theorem 2.2 and Proposition 4.3 one can formulate the following result.

Proposition 4.4.

(i) *The problem* (VO) *is stable with respect to* φ_2 *if and only if for each solution* $\bar{x}$ *to* (VO), *there exists a solution* $\bar{t} \in \mathbb{R}^n$ *to the dual problem* $(\widehat{D}_F^{VO})$ *such that* $f(\bar{x}) \in -f_p^*(\bar{t}) + (\min_{x\in G} \bar{t}^T x)_p$. *In this case* $\bar{t}^T \bar{x} = \min_{x\in G} \bar{t}^T x$.

(ii) *Conversely, if* $\bar{x} \in G$ *and* $\bar{t} \in \mathbb{R}^n$ *satisfy the relations above, then* $\bar{x}$ *and* $\bar{t}$ *are solutions to* (VO) *and* $(\widehat{D}_F^{VO})$, *respectively.*

Remark 4.2. Let $p = 1$ and the assumptions of Theorem 2.8 in [5] be fulfilled. Then Proposition 4.4 is nothing else than the result which provides the optimality conditions (cf. [5, Theorem 2.10]) for the Fenchel dual problem in scalar optimization.

Fenchel–Lagrange duality. The last dual problem in this section deals with the perturbation function φ_3.

Proposition 4.5. *Let* $t \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}^m$. *Assume that* $\max_{\mathbb{R}_+^p\setminus\{0\}}\{(\lambda^T g(x) - t^T x)_p \mid x \in X\} \neq \emptyset$. *Then* $\varphi_3^*(0,t,\lambda) = f_p^*(t) + (\max_{x\in X}[\lambda^T g(x) - t^T x])_p$, *if* $\lambda \underset{\mathbb{R}_+^m}{\leqslant} 0$ *and is equal* $\emptyset$, *otherwise.*

Proof. Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$. By definition

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(t^T v\right)_p + \left(\lambda^T u\right)_p - \varphi_3(x,v,u) \;\middle|\; x\in\mathbb{R}^n, v\in\mathbb{R}^n, u\in\mathbb{R}^m \right\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(t^T v\right)_p + \left(\lambda^T u\right)_p - f(x+v) \;\middle|\; x\in X, v\in\mathbb{R}^n, g(x) \leqq_{\mathbb{R}^m_+} u \right\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \left\{ \left(t^T v\right)_p + \left(\lambda^T u\right)_p - f(x+v) \;\middle|\; g(x) \leqq_{\mathbb{R}^m_+} u \right\}.
\end{aligned}$$

Taking $\bar{u} := u - g(x)$, one has

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \left\{ \left(t^T v\right)_p + \left(\lambda^T g(x)\right)_p + \left(\lambda^T \bar{u}\right)_p - f(x+v) \;\middle|\; \bar{u}\in\mathbb{R}^m_+ \right\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \left\{ \left(t^T v\right)_p + \left(\lambda^T g(x)\right)_p - f(x+v) + \left\{ \left(\lambda^T \bar{u}\right)_p \;\middle|\; \bar{u}\in\mathbb{R}^m_+ \right\} \right\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \left\{ \left(\lambda^T g(x)\right)_p + \left\{ \left(\lambda^T \bar{u}\right)_p \;\middle|\; \bar{u}\in\mathbb{R}^m_+ \right\} \right. \\
&\qquad \left. + \left\{ \left(t^T v\right)_p - f(x+v) \;\middle|\; v\in\mathbb{R}^n \right\} \right\}.
\end{aligned}$$

Setting now $\bar{v} := x + v$, it follows that

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \left\{ \left(\lambda^T g(x)\right)_p + \left\{ \left(\lambda^T \bar{u}\right)_p \;\middle|\; \bar{u}\in\mathbb{R}^m_+ \right\} \right. \\
&\qquad \left. + \left\{ \left(t^T \bar{v}\right)_p - \left(t^T x\right)_p - f(\bar{v}) \;\middle|\; \bar{v}\in\mathbb{R}^n \right\} \right\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \left\{ \left(\lambda^T g(x)\right)_p - \left(t^T x\right)_p + \left\{ \left(\lambda^T \bar{u}\right)_p \;\middle|\; \bar{u}\in\mathbb{R}^m_+ \right\} \right. \\
&\qquad \left. + \left\{ \left(t^T \bar{v}\right)_p - f(\bar{v}) \;\middle|\; \bar{v}\in\mathbb{R}^n \right\} \right\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left\{ \left(\lambda^T u\right)_p \;\middle|\; u\in\mathbb{R}^m_+ \right\} + \left\{ \left(t^T v\right)_p - f(v) \;\middle|\; v\in\mathbb{R}^n \right\} \right. \\
&\qquad \left. + \left\{ \left(\lambda^T g(x)\right)_p - \left(t^T x\right)_p \;\middle|\; x\in X \right\} \right\}.
\end{aligned}$$

Consequently one has

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(\lambda^T u\right)_p \;\middle|\; u\in\mathbb{R}^m_+ \right\} \\
&\qquad + \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(t^T v\right)_p - f(\bar{v}) \;\middle|\; v\in\mathbb{R}^n \right\} + \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(\lambda^T g(x)\right)_p - \left(t^T x\right)_p \;\middle|\; x\in X \right\}.
\end{aligned}$$

Moreover, we can easy verify that

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left(\lambda^T g(x) - t^T x\right)_p \;\middle|\; x\in X \right\} = \left\{ \left(\max_{x\in X} \left[\lambda^T g(x) - t^T x\right] \right)_p \right\}.$$

By Lemma 4.1 we conclude that $\varphi_3^*(0,t,\lambda) \subseteq f_p^*(t) + (\max_{x\in X}[\lambda^T g(x) - t^T x])_p$, if $\lambda \leqq_{\mathbb{R}^m_+} 0$, being equal $\emptyset$, otherwise.

Let us show now the converse inclusion. Let $t\in\mathbb{R}^n$, $\lambda \leqq_{\mathbb{R}^m_+} 0$, $\bar{y} \in f_p^*(t) + (\max_{x\in\mathbb{R}^n}[\lambda^T g(x) - t^T x])_p$. Then it holds

$$\bar{y} \in \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \left\{ \left(t^T v\right)_p - f(v) \;\middle|\; v\in\mathbb{R}^n \right\} + \left(\max_{x\in X} \left[\lambda^T g(x) - t^T x\right] \right)_p \right\}.$$

In other words $\bar{y} \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleqslant} (t^T v)_p - f(v) + (\max_{x\in X}[\lambda^T g(x) - t^T x])_p$, $\forall v \in \mathbb{R}^n$. Since

$$\big(t^T v\big)_p - f(v) + \big(\lambda^T g(x) - t^T x\big)_p \underset{\mathbb{R}^p_+}{\leqslant} \big(t^T v\big)_p - f(v) + \Big(\max_{x\in X}\big[\lambda^T g(x) - t^T x\big]\Big)_p,$$
$$\forall x \in X,$$

we conclude that $\bar{y} \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleqslant} (t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p,\ \forall x \in X,\ \forall v \in \mathbb{R}^n$ or, equivalently,

$$\bar{y} + \big(\lambda^T u\big)_p \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleqslant} \big(t^T v\big)_p - f(v) + \big(\lambda^T g(x) - t^T x\big)_p + \big(\lambda^T u\big)_p,$$
$$\forall x \in X,\ \forall v \in \mathbb{R}^n,\ \forall u \in \mathbb{R}^m_+.$$

On the other hand, because of $(\lambda^T u)_p \underset{\mathbb{R}^p_+}{\leqslant} 0$, $\forall u \in \mathbb{R}^m_+$ it holds $\bar{y} \underset{\mathbb{R}^p_+}{\geqslant} \bar{y} + (\lambda^T u)_p$, $u \in \mathbb{R}^m_+$. Whence, we obtain that $\bar{y} \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleqslant} (t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p + (\lambda^T u)_p$, $\forall x \in X$, $\forall v \in \mathbb{R}^n$, $\forall u \in \mathbb{R}^m_+$. Therefore $\bar{y} \in \varphi_3^*(0, t, \lambda)$. □

Now we can define the following Fenchel–Lagrange-type dual problem to (VO)

$$\big(\widehat{D}^{VO}_{FL}\big) \quad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(t,\lambda)\in\mathbb{R}^n\times\mathbb{R}^m} \big[-\varphi_3^*(0, t, \lambda)\big]$$
$$= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\substack{t\in\mathbb{R}^n \\ \lambda \underset{\mathbb{R}^m_+}{\leqslant} 0}} \Big[-f_p^*(t) + \Big(\min_{x\in X}\big[t^T x - \lambda^T g(x)\big]\Big)_p\Big]$$
$$= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\substack{t\in\mathbb{R}^n \\ \lambda \underset{\mathbb{R}^m_+}{\geqslant} 0}} \Big[-f_p^*(t) + \Big(\min_{x\in X}\big[t^T x + \lambda^T g(x)\big]\Big)_p\Big].$$

According to Theorem 2.2 and Proposition 4.5 one can state the following result.

Proposition 4.6.

(i) *The problem (VO) is stable with respect to φ_3 if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{t} \in \mathbb{R}^n$, $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geqslant} 0$ to the dual problem $(\widehat{D}^{VO}_{FL})$ such that $f(\bar{x}) \in -f_p^*(\bar{t}) + (\min_{x\in X}[\bar{t}^T x + \bar{\lambda}^T g(x)])_p$. In this case $\bar{t}^T \bar{x} + \bar{\lambda}^T g(\bar{x}) = \min_{x\in X}[\bar{t}^T x + \bar{\lambda}^T g(x)]$ and $\bar{\lambda}^T g(\bar{x}) = 0$.*

(ii) *Conversely, if $\bar{x} \in G$ and $\bar{t} \in \mathbb{R}^n$, $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geqslant} 0$ satisfy the relations above, then $\bar{x}$ and $(\bar{t}, \bar{\lambda})$ are solutions to (VO) and $(\widehat{D}^{VO}_{FL})$, respectively.*

Remark 4.3. In the scalar case Proposition 4.6 is nothing else than the assertion dealing with the optimality conditions for the Fenchel–Lagrange duality (cf. [5, Theorem 2.11]).

Further we show some relations between the dual objective maps investigated in this section.

Proposition 4.7. *Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$ with $\lambda \leqslant_{\mathbb{R}^m_+} 0$. If $\max_{\mathbb{R}^p_+\setminus\{0\}}\{(-t^T x)_p \mid x \in G\} \neq \emptyset$ and $\max_{\mathbb{R}^p_+\setminus\{0\}}\{(\lambda^T g(x) - t^T x)_p \mid x \in X\} \neq \emptyset$, then $\varphi_2^*(0,t) \subseteq \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+$.*

Proof. Let $t \in \mathbb{R}^n$ and $\lambda \leqslant_{\mathbb{R}^m_+} 0$. Assume that $z \in \varphi_2^*(0,t) = f_p^*(t) - (\min_{x\in G} t^T x)_p$. Since $g(x) \leqslant_{\mathbb{R}^m_+} 0$, for $x \in G$ one has $-\lambda^T g(x) \leqslant 0$, $\forall x \in G$. After adding $t^T x$ in both sides we have

$$\min_{x\in X}\bigl[t^T x - \lambda^T g(x)\bigr] \leqslant \min_{x\in G}\bigl[t^T x - \lambda^T g(x)\bigr] \leqslant \min_{x\in G} t^T x$$

and so

$$-\Bigl(\min_{x\in G} t^T x\Bigr)_p \leqslant_{\mathbb{R}^p_+} -\Bigl(\min_{x\in X}\bigl[t^T x - \lambda^T g(x)\bigr]\Bigr)_p.$$

This means that

$$-\Bigl(\min_{x\in G} t^T x\Bigr)_p \in -\Bigl(\min_{x\in X}\bigl[t^T x - \lambda^T g(x)\bigr]\Bigr)_p - \mathbb{R}^p_+.$$

Therefore $z \in f_p^*(t) - (\min_{x\in X}[t^T x - \lambda^T g(x)])_p - \mathbb{R}^p_+$. In other words $z \in \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+$. □

Proposition 4.8. *Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$ with $\lambda \leqslant_{\mathbb{R}^m_+} 0$. If the set $f_p^*(t)$ is external stable and $\max_{\mathbb{R}^p_+\setminus\{0\}}\{(\lambda^T g(x) - t^T x)_p \mid x \in X\} \neq \emptyset$, then $\varphi_1^*(0,\lambda) \subseteq \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+$.*

Proof. Let $t \in \mathbb{R}^n$ and $\lambda \leqslant_{\mathbb{R}^m_+} 0$ be fixed. Then one has

$$\begin{aligned}\varphi_1^*(0,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}}\bigl\{\bigl(\lambda^T g(x)\bigr)_p - f(x) \bigm| x \in X\bigr\} \subseteq \bigl\{\bigl(\lambda^T g(x)\bigr)_p - f(x) \bigm| x \in X\bigr\} \\ &\subseteq \bigl\{\bigl(t^T x\bigr)_p - f(x) \bigm| x \in \mathbb{R}^n\bigr\} + \bigl\{-\bigl(t^T x - \lambda^T g(x)\bigr)_p \bigm| x \in X\bigr\}.\end{aligned}$$

On the other hand, in view of the relation $-\{(p^T x - \lambda^T g(x))_p \mid x \in X\} \subseteq -\min_{x\in X}(p^T x - \lambda^T g(x))_p - \mathbb{R}^p_+$ and by the external stability of $f_p^*(t)$, we have

$$\begin{aligned}\varphi_1^*(0,\lambda) &\subseteq f_p^*(t) - \mathbb{R}^p_+ - \min_{x\in X}\bigl(p^T x - \lambda^T g(x)\bigr)_p - \mathbb{R}^p_+ \\ &= f_p^*(t) - \min_{x\in X}\bigl(p^T x - \lambda^T g(x)\bigr)_p - \mathbb{R}^p_+ = \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+. \qquad \square\end{aligned}$$

5. Applications to vector variational inequalities

5.1. Gap functions for vector variational inequalities

Let $F : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ be a matrix-valued function and $K \subseteq \mathbb{R}^n$. The vector variational inequality problem consists in finding $x \in K$ such that

$$(VVI) \quad F(x)^T(y - x) \nleq_{\mathbb{R}^p_+\setminus\{0\}} 0, \quad \forall y \in K.$$

Definition 5.1. (Cf. [6,11].) A set-valued map $\gamma : K \rightrightarrows \mathbb{R}^p$ is said to be a gap function for (*VVI*) if it satisfies the following conditions:

(i) $0 \in \gamma(x)$ if and only if $x \in K$ solves the problem (*VVI*);
(ii) $0 \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\geqslant} \gamma(y),\ \forall y \in K$.

For (*VVI*) the following gap function has been introduced in the past (see [6])

$$\gamma_A^{VVI}(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \big\{F(x)^T(x-y) \bigm| y \in K\big\}.$$

Let us notice that γ_A^{VVI} is a generalization of Auslender's gap function for the scalar variational inequality problem (cf. [4]). On the other hand, the duality results investigated in Section 3 allow us to introduce some new gap functions for (*VVI*). Let us mention that such a similar approach has been used for scalar variational inequalities in [3]. We remark that $x \in K$ is a solution to the problem (*VVI*) if and only if 0 is a minimal point of the set $\{F(x)^T(y-x) \mid y \in K\}$. This means that x is a solution of the following vector optimization problem

$$\big(P^{VVI}; x\big) \quad \min_{\mathbb{R}^p_+\setminus\{0\}} \big\{F(x)^T(y-x) \bigm| y \in K\big\}.$$

Let $x \in K$ be fixed. Setting $\tilde{f}_x(y) := F(x)^T(y-x)$ instead of f in (D_F^{VO}), the Fenchel dual problem to $(P^{VVI}; x)$ turns out to be

$$\big(D_F^{VVI}; x\big) \quad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \min_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \bigcup_{y\in\mathbb{R}^n} \big\{\big(F(x)^T - T\big)y\big\} - F(x)^T x + \bigcup_{y\in K}\{Ty\} \right\}.$$

We define the following map for any $x \in K$, $\gamma_F^{VVI}(x) := \bigcup_{T\in\mathbb{R}^{p\times n}} \widetilde{\Phi}_2^*(0,T;x)$, where $\widetilde{\Phi}_2^*(0,T;x)$ is defined by

$$\widetilde{\Phi}_2^*(0,T;x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \bigcup_{y\in\mathbb{R}^n} \big\{\big(T - F(x)^T\big)y\big\} + F(x)^T x + \bigcup_{y\in K}\{-Ty\} \right\}.$$

Theorem 5.1. *Let for any $x \in K$ the problem $(P^{VVI}; x)$ be stable with respect to $\widetilde{\Phi}_2(0,\cdot;x)$. Then γ_F^{VVI} is a gap function for (VVI).*

Proof. (i) Let $x \in K$ be a solution to the problem (*VVI*). As the problem $(P^{VVI}; x)$ is stable, by Proposition 3.10(i), there exists a solution $T_x \in \mathbb{R}^{p\times n}$ to $(D_F^{VVI}; x)$ such that $\tilde{f}_x(x) = 0 \in -\widetilde{\Phi}_2^*(0,T_x;x)$. In other words, $0 \in \widetilde{\Phi}_2^*(0,T_x;x)$ and this implies that $0 \in \bigcup_{T\in\mathbb{R}^{p\times n}} \widetilde{\Phi}_2^*(0,T;x) = \gamma_F^{VVI}(x)$. Conversely, let $x \in K$ and $0 \in \gamma_F^{VVI}(x)$. Hence, there exists $T_x \in \mathbb{R}^{p\times n}$ such that $0 \in \widetilde{\Phi}_2^*(0,T_x;x)$ or, equivalently, $0 = F(x)^T(x-x) \in -\widetilde{\Phi}_2^*(0,T_x;x)$. According to Proposition 3.10(ii), x is a solution to $(P^{VVI}; x)$ and also to the problem (*VVI*).

(ii) Let $y \in K$ be fixed. Then, in view of Proposition 3.9, for any $T \in \mathbb{R}^{p\times n}$, one has $\tilde{f}_y(z) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleqslant} 0,\ \forall z \in K,\ \forall \xi \in \widetilde{\Phi}_2^*(0,T;y)$ or, equivalently, $F(y)^T(z-y) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleqslant} 0,\ \forall z \in K,\ \forall \xi \in \bigcup_{T\in\mathbb{R}^{p\times n}} \widetilde{\Phi}_2^*(0,T;y) = \gamma_F^{VVI}(y)$. Setting $z = y$, we get $\xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleqslant} 0,\ \forall \xi \in \gamma_F^{VVI}(y)$. □

According to Proposition 3.2, we can give the following result relative to the stability with respect to $\widetilde{\Phi}_2(0,\cdot;x)$ when $x \in K$.

Proposition 5.1. *Let for any $x \in K$ the set $\min_{\mathbb{R}^p_+\setminus\{0\}}\{F(x)^T y \mid y \in K\}$ be externally stable. Then the problem $(P^{VVI}; x)$ is stable with respect to $\widetilde{\Phi}_2(0, \cdot; x)$.*

In connection with the Fenchel dual problem we call γ_F^{VVI} the *Fenchel gap function* for the problem (*VVI*). Let now the ground set K be given by $K = \{x \in \mathbb{R}^n \mid g(x) \leqslant_{\mathbb{R}^m_+} 0\}$, where $g(x) = (g_1(x), \ldots, g_m(x))^T$, $g_i : \mathbb{R}^n \to \mathbb{R}$. Before introducing two other gap functions, let us formulate the Lagrange and Fenchel–Lagrange dual problems for $(P^{VVI}; x)$. Taking $\tilde{f}_x$ instead of f in $\Phi_1^*(0, \Lambda)$ and $\Phi_3^*(0, T, \Lambda)$, respectively, we have

$$\left(D_L^{VVI}; x\right) \quad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda \in \mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \bigcup_{q \in \mathbb{R}^m_+} \{-\Lambda q\} - F(x)^T x + \bigcup_{y \in \mathbb{R}^n} \left\{F(x)^T y - \Lambda g(y)\right\} \right\}$$

and

$$\begin{aligned} \left(D_{FL}^{VVI}; x\right) \quad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda) \in \mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \bigcup_{q \in \mathbb{R}^m_+} \{-\Lambda q\} - F(x)^T x \right. \\ & \left. + \bigcup_{y \in \mathbb{R}^n} \left\{\left(F(x)^T - T\right)y\right\} + \bigcup_{y \in \mathbb{R}^n} \left\{Ty - \Lambda g(y)\right\} \right\}. \end{aligned}$$

For $x \in K$, we define the gap functions as follows $\gamma_L^{VVI}(x) := \bigcup_{\Lambda \in \mathbb{R}^{p\times m}} \widetilde{\Phi}_1^*(0, \Lambda; x)$, where

$$\widetilde{\Phi}_1^*(0, \Lambda; x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \bigcup_{q \in \mathbb{R}^m_+} \{\Lambda q\} + F(x)^T x + \bigcup_{y \in \mathbb{R}^n} \left\{\Lambda g(y) - F(x)^T y\right\} \right\}$$

and

$$\gamma_{FL}^{VVI}(x) := \bigcup_{(T,\Lambda) \in \mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \widetilde{\Phi}_3^*(0, T, \Lambda; x),$$

where

$$\begin{aligned} & \widetilde{\Phi}_3^*(0, T, \Lambda; x) \\ & \quad = \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \bigcup_{q \in \mathbb{R}^m_+} \{\Lambda q\} + F(x)^T x + \bigcup_{y \in \mathbb{R}^n} \left\{\left(T - F(x)^T\right)y\right\} + \bigcup_{y \in \mathbb{R}^n} \left\{\Lambda g(y) - Ty\right\} \right\}, \end{aligned}$$

respectively. In analogy to the proof of Theorem 5.1, by applying the duality assertions in Section 3 for (D_L^{VO}) and (D_{FL}^{VO}), respectively, the following theorem can be verified.

Theorem 5.2. *Let for any $x \in K$ the problem $(P^{VVI}; x)$ be stable with respect to $\widetilde{\Phi}_1(0, \cdot; x)$ and $\widetilde{\Phi}_3(0, \cdot; x)$, respectively. Then γ_L^{VVI} and γ_{FL}^{VVI} are gap functions for (VVI).*

The origin of these new gap functions for (*VVI*) justifies to call them the *Lagrange gap function γ_L^{VVI}* and the *Fenchel–Lagrange gap function γ_{FL}^{VVI}*, respectively.

5.2. Gap functions via Fenchel duality

According to the results in Section 4, we can suggest a further class of gap functions to (*VVI*). In this subsection, we restrict the construction of a gap function to the case of Fenchel duality. As mentioned before, for a fixed $x \in K$ we consider the following vector optimization problem relative to (*VVI*)

$$\left(P^{VVI}; x\right) \qquad \min_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ F(x)^T (y - x) \;\middle|\; y \in K \right\}.$$

For a fixed $x \in K$, taking $F(x)^T(y-x)$ as the objective function, $(\widehat{D}^{VO}_F)$ becomes

$$\left(\widehat{D}^{VVI}_F; x\right) \quad \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in \mathbb{R}^n} \left\{ \min_{\mathbb{R}^p_+ \setminus \{0\}} \left[F(x)^T (y - x) - \left(t^T y\right)_p \;\middle|\; y \in \mathbb{R}^n \right] + \left(\min_{y \in K} t^T y \right)_p \right\}.$$

We need first the following auxiliary result.

Lemma 5.1. *Let $M \in \mathbb{R}^{p\times n}$. Then $\min_{\mathbb{R}^p_+ \setminus \{0\}} \{My \mid y \in \mathbb{R}^n\} = \{My \mid y \in \mathbb{R}^n\}$, if exists $\mu \in \operatorname{int} \mathbb{R}^p_+$ such that $\mu^T M = 0$ and is equal $\emptyset$, otherwise.*

Proof. Let $M \in \mathbb{R}^{p\times n}$ be fixed and $\bar{y} \in \mathbb{R}^n$. According to Theorem 11.20 in [12], $M\bar{y} \in \min_{\mathbb{R}^p_+ \setminus \{0\}} \{My \mid y \in \mathbb{R}^n\}$ if and only if $\exists \mu \in \operatorname{int} \mathbb{R}^p_+$ such that $\mu^T M \bar{y} \leqslant \mu^T M y$, $\forall y \in \mathbb{R}^n$. As $\inf_{y\in\mathbb{R}^n} \mu^T M y = 0$, if $\mu^T M = 0$ and is equal $-\infty$, otherwise, the conclusion follows. □

Let $C := [t, \dots, t] \in \mathbb{R}^{n\times p}$ and for a fixed $x \in K$ let $N(x)$ be the set defined by

$$N(x) := \left\{ t \in \mathbb{R}^n \;\middle|\; \exists \mu \in \operatorname{int} \mathbb{R}^p_+ \text{ such that } (F(x) - C)\mu = 0 \right\}.$$

In view of Lemma 5.1, the dual becomes

$$\left(\widehat{D}^{VVI}_F; x\right) \quad \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in N(x)} \left\{ -F(x)^T x + \left\{ \left(F(x) - C\right)^T y \;\middle|\; y \in \mathbb{R}^n \right\} + \left(\min_{y \in K} t^T y \right)_p \right\}.$$

We introduce for $x \in K$ the following map

$$\tilde{\gamma}^{VVI}_F(x) := F(x)^T x + \bigcup_{t \in N(x)} \left[\left\{ \left(C - F(x)\right)^T y \;\middle|\; y \in \mathbb{R}^n \right\} - \left(\min_{y \in K} t^T y \right)_p \right].$$

Theorem 5.3. *Let for any $x \in K$ the set $\min_{\mathbb{R}^p_+ \setminus \{0\}} \{F(x)^T y \mid y \in K\}$ be externally stable. Then $\tilde{\gamma}^{VVI}_F$ is a gap function for (VVI).*

Proof. (i) Let $x \in K$ be fixed. As the set $\min_{\mathbb{R}^p_+ \setminus \{0\}} \{F(x)^T y \mid y \in K\}$ is externally stable, by Proposition 5.1, the problem $(P^{VVI}; x)$ is stable. Taking $F(x)^T(y-x)$ instead of $f(y)$ in the formula of $f^*_p(t)$, by Lemma 5.1, this becomes

$$\begin{aligned} &\max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(t^T y\right)_p - F(x)^T (y - x) \;\middle|\; y \in \mathbb{R}^n \right\} \\ &\quad = F(x)^T x - \min_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ \left(F(x) - C\right)^T y \;\middle|\; y \in \mathbb{R}^n \right\} = F(x)^T x - \left\{ \left(F(x) - C\right)^T y \;\middle|\; y \in \mathbb{R}^n \right\}, \end{aligned}$$

where $C = [t, \dots, t] \in \mathbb{R}^{n\times p}$ and $t \in N(x)$.

Let $\bar{x} \in K$ be a solution to (VVI). By Proposition 4.4(i) it follows that

$$0 \in -F(x)^T x + \left\{ \left(F(x) - C\right)^T y \mid y \in \mathbb{R}^n \right\} + \left(\min_{y \in K} t^T y \right)_p$$

and so $0 \in \tilde{\gamma}_F^{VVI}(\bar{x})$. Conversely, let $\bar{x} \in K$ and $0 \in \tilde{\gamma}_F^{VVI}(\bar{x})$. Then $\exists \bar{t} \in N(\bar{x})$ such that $0 \in F(\bar{x})^T \bar{x} + \{(\bar{C} - F(\bar{x}))^T y \mid y \in \mathbb{R}^n\} - (\min_{y \in K} \bar{t}^T y)_p$, where $\bar{C} = [\bar{t}, \dots, \bar{t}] \in \mathbb{R}^{n \times p}$. Taking into account Proposition 4.4(ii), $\bar{x}$ is a solution to $(P^{VVI}; \bar{x})$. Consequently, $\bar{x}$ solves the problem (VVI).

(ii) Let $y \in K$. Choosing $T := [t, \dots, t]^T \in \mathbb{R}^{p \times n}$, by Propositions 3.9 and 4.3, it holds $F(y)^T (z - y) + \xi \nleq_{\mathbb{R}^p_+ \setminus \{0\}} 0$, $\forall z \in K$, $\forall \xi \in f_p^*(t) - (\min_{y \in K} t^T y)_p$, $\forall t \in N(y)$ or, equivalently, $F(y)^T (z - y) + \xi \nleq_{\mathbb{R}^p_+ \setminus \{0\}} 0$, $\forall z \in K$, $\forall \xi \in \tilde{\gamma}_F^{VVI}(y)$. Setting $z = y$, one has $\xi \nleq_{\mathbb{R}^p_+ \setminus \{0\}} 0$, $\forall \xi \in \tilde{\gamma}_F^{VVI}(y)$. $\square$

Remark 5.1. In the case $p = 1$, the problem (VVI) reduces to the scalar variational inequality problem of finding $x \in K$ such that

$$(VI) \quad F(x)^T (x - y) \geqslant 0, \quad y \in K,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function. Let $x \in K$ be fixed. If $t \in N(x)$, there exists $\mu > 0$ such that $(F(x) - t)\mu = 0$. Therefore it holds $F(x) = t$. Consequently, the gap function for the variational inequality (VI) becomes $\gamma_F^{VI}(x) = F(x)^T x + \max_{y \in K}(-F(x)^T y) = \max_{y \in K} F(x)^T (x - y)$, which coincides with Auslender's gap function (see [3,4]).

Example 5.1. Let $F = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be a constant matrix and $K = \{(x_1, x_2)^T \in \mathbb{R}^2 \mid 0 \leqslant x_i \leqslant 1,\ x_i \in \mathbb{R},\ i = 1, 2\}$. We consider the vector variational inequality problem of finding $x \in K$ such that

$$(VVI_1) \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (y - x) \nleq_{\mathbb{R}^2_+ \setminus \{0\}} 0, \quad \forall y \in K.$$

We calculate $\tilde{\gamma}_F^{VVI}$ for (VVI_1). Let $x = (x_1, x_2)^T \in \mathbb{R}^2$ be fixed. First we consider the set-valued map $W : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ given by (see $(\widehat{D}_F^{VVI}; x)$) $W(x_1, x_2) = \min_{\mathbb{R}^2_+ \setminus \{0\}} \{F(x)^T (y - x) - (t^T y)_2 \mid y \in \mathbb{R}^2\}$. Then

$$W(x_1, x_2) = \min_{\mathbb{R}^2_+ \setminus \{0\}} \left\{ \begin{pmatrix} (1 - t_1) y_1 - t_2 y_2 \\ -t_1 y_1 + (1 - t_2) y_2 \end{pmatrix} \,\middle|\, (y_1, y_2)^T \in \mathbb{R}^2 \right\} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

If $\exists \mu = (\mu_1, \mu_2)^T \in \operatorname{int} \mathbb{R}^2_+$ such that $(1 - t_1)\mu_1 - t_1 \mu_2 = 0$ and $-t_2 \mu_1 + (1 - t_2)\mu_2 = 0$ or, equivalently, $t_1 + t_2 = 1$ and $t_2 \mu_1 = t_1 \mu_2$, then, by Lemma 5.1, it holds

$$W(x_1, x_2) = \left\{ \begin{pmatrix} (1 - t_1) y_1 - t_2 y_2 \\ -t_1 y_1 + (1 - t_2) y_2 \end{pmatrix} \,\middle|\, (y_1, y_2)^T \in \mathbb{R}^2 \right\} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Whence

$$\tilde{\gamma}_F^{VVI_1}(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \bigcup_{t \in N_1} \left[\left\{ \begin{pmatrix} t(y_2 - y_1) \\ (1 - t)(y_1 - y_2) \end{pmatrix} \,\middle|\, (y_1, y_2)^T \in \mathbb{R}^2 \right\} \right.$$
$$\left. - \begin{pmatrix} \min_{0 \leqslant y_1 \leqslant 1} (1 - t) y_1 + \min_{0 \leqslant y_2 \leqslant 1} t y_2 \\ \min_{0 \leqslant y_1 \leqslant 1} (1 - t) y_1 + \min_{0 \leqslant y_2 \leqslant 1} t y_2 \end{pmatrix} \right],$$

where the set N_1 is defined by $N_1 := \{t \in \mathbb{R} \mid \exists \mu \in \operatorname{int} \mathbb{R}^2_+ \text{ such that } (1-t)\mu_1 = t\mu_2\}$. Further, as $N_1 = (0,1)$, we obtain that

$$\tilde{\gamma}_F^{VVI_1}(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \bigcup_{t \in (0,1)} \left\{ \begin{pmatrix} ty \\ (t-1)y \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}.$$

Remark 5.2. A very interesting problem arises when one wants to obtain error bounds for the solutions of the vector variational inequality (*VVI*) by using the gap functions introduced in this section like Noor did in [18] for the general variational inequality. As the gap functions are set-valued functions, the real challenge is to find some scalar gap functions attached to them which characterize the solutions of (*VVI*) in a similar way. Then one could try to obtain the error bounds for the solutions of (*VVI*) by means of these intermediate scalar functions.

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